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Dilemma and quantum battle of sexes

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Abstract
We analysed quantum version of the game battle of sexes using a general initial quantum state. For a particular choice of initial entangled quantum state it is shown that the classical dilemma of the battle of sexes can be resolved and a unique solution of the game can be obtained.

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1. Introduction

Game theory deals with situations where two or more players compete to maximize their respective payoff or gain. Their interaction is strategic in nature as their gain depends on the strategy adopted by the other player/players [1]. In games with complete information, players decide their strategy on the basis of a payoff matrix known to them. Nash equilibrium (NE) is a key concept, which is a set of strategies from which unilateral deviation by a player reduces his/her payoff [2].

In an important two-player game, battle of sexes, Alice and Bob try to decide a place to spend a Saturday evening together. Alice wants to go to the opera and Bob is interested in watching TV while both prefer to spend their evening together. The two NE of the game correspond to situations when both the players choose either opera or TV. Alice prefers NE based on opera and Bob prefers the other NE. In the absence of any communication the two players face a dilemma in choosing between the two NE and could inadvertently end up with mismatched strategies. This mismatched strategy results in a loss for both players as they will not be able to spend the evening together, which is termed as the worst payoff for both the players.

Extension of game theory to the quantum domain [3] leads to some interesting new results in addition to the resolution of some existing dilemmas in the classical versions of games. In quantum game theory an initial quantum state is prepared by the arbiter and passed on to the players. After applying their respective local operators (or strategies) they return it to the arbiter who then determines their payoffs. The initial quantum state plays a crucial role and interesting results are obtained for initially entangled quantum states. Since players are equipped with local operators they are unable to determine the complete initial quantum state given to them and hence choose their strategies to maximize their payoff on the basis of the payoff matrix known to them. In an interesting example, Eisert *et al* [4] examined the game prisoner dilemma in the quantum domain and showed that the dilemma which exists in the classical version of the game does not exist in the quantum version. Further they constructed a quantum strategy which always wins over any classical strategy. Inspired by their work, Marinatto and Weber [5] proposed another interesting scheme to quantize the game of battle of sexes. They introduced Hilbert structure to the strategic space of the game and argued that if the players are allowed to play quantum strategies involving unitary operators then the game has a unique solution, and the dilemma could be resolved. They used maximally entangled initial quantum state, and allowed the players to play strategies which are combination of the identity operator I and the flip operator C, with classical probabilities p^* and $(1-p^*)$ for Alice and q^* and $(1-q^*)$ for Bob. In the quantum version of the game, maximum payoff is obtained for the two pure NE,($p^* = q^* = 1$) or $(1, 1)$ and $(0, 0)$. In the Marinatto and Weber scheme both NE correspond to equal payoff and hence they argued that the classical dilemma no longer exists as both the solutions or NE are equally good for the two players.

In an interesting comment Benjamin [6] pointed out that the dilemma still exists as the same payoff for the two NE make them equally acceptable to both the players and there is no way for them to prefer '1' over '0' , in the absence of any communication between them. He argued that the players still face a somewhat similar dilemma as they could still end up with a situation $(1, 0)$ or $(0, 1)$ which corresponds to the worst payoff for both. In their response to Benjamin's comment, Marinatto and Weber [7] insisted that since both the NE $(0, 0)$ and $(1, 1)$ render the initial quantum state unchanged and correspond to equal and maximum payoff for both the players, therefore both the players would prefer $(1, 1)$, as by choosing p or q equal to zero there is a danger for both the players of getting into a situation $(1, 0)$ or $(0, 1)$ which corresponds to the lowest payoff. However, choosing a strategy on this argument requires complete information on the initial quantum state and in quantum games players cannot measure the initial quantum state [8–10].

In this paper we consider a general initial quantum state and present a condition on the parameters of the initial quantum state for which the classical dilemma can be resolved and a unique solution of the quantum battle of sexes can be obtained. In comparison with Marinatto and Weber [5] we present a condition for which payoffs corresponding to 'mismatched or worst-case situation' are different for the two players, which results in a unique solution of the game. Classical and quantum versions of the game of battle of sexes are presented in section 2. Here we restrict our analysis to pure strategies only as the classical dilemma deals with these strategies only.

2. Battle of sexes

2.1. Classical form

Battle of sexes is an interesting static game. In the usual exposition of this game two players Alice and Bob try to decide a place to spend a Saturday evening together. Alice wants to go to the opera while Bob is interested in watching TV and both would like to be together. The game is represented by the following payoff matrix:

$$\begin{array}{cc} & \text{Bob} \\ & \begin{array}{cc} O & T \end{array} \\ \text{Alice} \begin{array}{c} O \\ T \end{array} & \begin{bmatrix} (\alpha, \beta) & (\gamma, \gamma) \\ (\gamma, \gamma) & (\beta, \alpha) \end{bmatrix} \end{array} \tag{1}$$

where O and T represent opera and TV, respectively. The elements α, β and γ are the payoffs for the players corresponding to the choices available to them with $\alpha > \beta > \gamma$. The two pure NE of this game are (O, O) and (T, T) which corresponds to the situation when both the players choose opera and TV, respectively. Here the first NE is more favourable to Alice while the second NE is favourable to Bob. Since the players are not allowed to communicate, they face a dilemma in choosing their strategy. The strategy pairs (O, T) and (T, O) correspond to the worst-case payoff for the two players, i.e., both the players get the minimum possible payoff γ. There also exists a mixed NE for this game but we are not interested in it here.

2.2. *Quantum form*

In the quantum version of the game both players, Alice and Bob, apply their respective strategies to the initial quantum state given to them on the basis of the payoff matrix known to them. In this approach the payoff matrix depends on the initial state and can be controlled by the parameters of the initial quantum state. Our choice of a general initial quantum state provides us with additional parameters to control the game in comparison with Marinatto and Weber's [5].

Let Alice and Bob have the following initial entangled quantum state at their disposal:

$$|\psi_{\text{in}}\rangle = a|OO\rangle + b|OT\rangle + c|TO\rangle + d|TT\rangle \tag{2}$$

where $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. Here the first entry in ket $|\,\rangle$ is for Alice and the second for Bob's strategy. For b and c equal to zero, equation (2) reduces to the initial entangled quantum state used by Marinatto and Weber [5]. The unitary operators at the disposal of the two players are defined as

$$\begin{aligned} &C|O\rangle = |T\rangle \qquad C|T\rangle = |O\rangle \qquad C = C^{\dagger} = C^{-1} \\ &I|O\rangle = |O\rangle \qquad I|T\rangle = |I\rangle \qquad I = I^{\dagger} = I^{-1}. \end{aligned} \tag{3}$$

Following the Marinatto and Weber approach, take $pI + (1-p)C$ and $qI + (1-q)C$ as the strategies for the two players, respectively, with p and q being the classical probabilities for using the identity operator I. The final density matrix takes the form

$$\begin{aligned} \rho_f = {} & pqI_A \otimes I_B \rho_{\text{in}} I_A^{\dagger} \otimes I_B^{\dagger} + p(1-q)I_A \otimes C_B \rho_{\text{in}} I_A^{\dagger} \otimes C_B^{\dagger} \\ & + q(1-p)C_A \otimes I_B \rho_{\text{in}} C_A^{\dagger} \otimes I_B^{\dagger} + (1-p)(1-q)C_A \otimes C_B \rho_{\text{in}} C_A^{\dagger} \otimes C_B^{\dagger}. \end{aligned} \tag{4}$$

Here $\rho_{\text{in}} = |\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|$ which can be obtained through equation (2) (see the appendix). The corresponding payoff operators for Alice and Bob are

$$P_A = \alpha|OO\rangle\langle OO| + \beta|TT\rangle\langle TT| + \gamma(|OT\rangle\langle OT| + |TO\rangle\langle TO|) \tag{5}$$

$$P_B = \beta|OO\rangle\langle OO| + \alpha|TT\rangle\langle TT| + \gamma(|OT\rangle\langle OT| + |TO\rangle\langle TO|) \tag{6}$$

and the payoff functions (the mean values of these operators, i.e. $\$_A(p, q) = \text{Tr}[P_A \rho_f]$ and $\$_B(p, q) = \text{Tr}[P_B \rho_f]$) are obtained by using equations (3)–(6) and are given as

$$\$_A(p, q) = p[q\Omega + \Phi(|b|^2 - |d|^2) + \Lambda(|c|^2 - |a|^2)] + q[\Lambda(|b|^2 - |a|^2) + \Phi(|c|^2 - |d|^2)] + \Theta \tag{7}$$

$$\$_B(p,q) = q[p\Omega + \Phi(|b|^2 - |a|^2) + \Lambda(|c|^2 - |d|^2)] + p[\Lambda(|b|^2 - |d|^2) + \Phi(|c|^2 - |a|^2)] + \Theta. \tag{8}$$

In writing the above equations we have used

$$\begin{aligned} &\Omega = (\alpha + \beta - 2\gamma)(|a|^2 - |b|^2 - |c|^2 + |d|^2) \\ &\Phi = (\alpha - \gamma) \qquad \Lambda = (\beta - \gamma) \\ &\Theta = \alpha|d|^2 + \gamma|c|^2 + \gamma|b|^2 + \beta|a|^2. \end{aligned} \tag{9}$$

The NE of the game are found by solving the following two inequalities:

$$\begin{aligned} &\$_A(p^*, q^*) - \$_A(p, q^*) \geqslant 0 \\ &\$_B(p^*, q^*) - \$_B(p, q^*) \geqslant 0 \end{aligned} \tag{10}$$

that lead to the following two conditions, respectively:

$$(p^* - p)[q^*(\alpha + \beta - 2\gamma)(|a|^2 - |b|^2 - |c|^2 + |d|^2) + (\gamma - \beta)|a|^2 + (\alpha - \gamma)|b|^2 + (\beta - \gamma)|c|^2 + (\gamma - \alpha)|d|^2] \geqslant 0 \tag{11}$$

and

$$(q^* - q)[p^*(\alpha + \beta - 2\gamma)(|a|^2 - |b|^2 - |c|^2 + |d|^2) + (\gamma - \alpha)|a|^2 + (\alpha - \gamma)|b|^2 + (\beta - \gamma)|c|^2 + (\gamma - \beta)|d|^2] \geqslant 0. \tag{12}$$

The above two inequalities are satisfied if both the factors have the same sign. Here we are interested in solving the dilemma arising due to pure strategies, i.e. (1, 1) and (0, 0), therefore, we restrict ourselves to the following possible pure strategy pairs:

Case a. When $p^* = 0, q^* = 0$ then the inequalities (11) and (12), reduce to

$$\begin{aligned} &(\gamma - \beta)|a|^2 + (\alpha - \gamma)|b|^2 + (\beta - \gamma)|c|^2 + (\gamma - \alpha)|d|^2 < 0 \\ &(\gamma - \alpha)|a|^2 + (\alpha - \gamma)|b|^2 + (\beta - \gamma)|c|^2 + (\gamma - \beta)|d|^2] < 0. \end{aligned} \tag{13}$$

All those values of the initial quantum state parameters for which the above inequalities are satisfied, strategy pair (0, 0) is a Nash equilibrium. Here we consider a particular set of values for the initial state parameters for which a unique solution of the game can be found and hence the dilemma would be resolved, however, this choice is not unique. Let us take

$$|a|^2 = |d|^2 = |b|^2 = \tfrac{5}{16} \qquad |c|^2 = \tfrac{1}{16}. \tag{14}$$

The corresponding payoffs obtained from equations (7) and (8) are

$$\$_A(0,0) = \frac{5\alpha + 5\beta + 6\gamma}{16} \qquad \$_B(0,0) = \frac{5\alpha + 5\beta + 6\gamma}{16}. \tag{15}$$

Physically it means that for the NE (0, 0), the two players get equal payoff corresponding to the choice of initial state parameters given by equation (14).

Case (b). When $p^* = q^* = 1$, then the inequalities (11) and (12) become

$$\begin{aligned} &(\alpha - \gamma)|a|^2 + (\gamma - \beta)|b|^2 + (\gamma - \alpha)|c|^2 + (\beta - \gamma)|d|^2 > 0 \\ &(\beta - \gamma)|a|^2 + (\gamma - \beta)|b|^2 + (\gamma - \alpha)|c|^2 + (\alpha - \gamma)|d|^2 > 0. \end{aligned} \tag{16}$$

These inequalities are again satisfied for the choice of parameters given in equation (14) for the initial quantum state and the strategy pair (1, 1) is also a NE. The corresponding payoffs for the two players in this case are

$$\$_A(1,1) = \frac{5\alpha + 5\beta + 6\gamma}{16} \qquad \$_B(1,1) = \frac{5\alpha + 5\beta + 6\gamma}{16}. \tag{17}$$

For the mismatched strategy pairs, i.e. $(p^* = 0, q^* = 1)$ and $(p^* = 1, q^* = 0)$ the inequalities (11) and (12) are not satisfied for the choice of initial state parameters given by equation (14), hence these strategy pairs are not NE. However, it is interesting to note the corresponding payoffs for the two players, i.e.

$$\begin{aligned} \$_A(0,1) &= \frac{\alpha + 5\beta + 10\gamma}{16} \qquad & \$_B(0,1) &= \frac{5\alpha + \beta + 10\gamma}{16} \\ \$_A(1,0) &= \frac{5\alpha + \beta + 10\gamma}{16} \qquad & \$_B(1,0) &= \frac{\alpha + 5\beta + 10\gamma}{16}. \end{aligned} \tag{18}$$

Now keeping in view all the payoffs given by equations (15), (17), (18), under the choice (14), the quantum game can be represented by the following payoff matrix:

$$\begin{array}{ccc} & & \text{Bob} \\ & & \begin{array}{cc} q=1 & q=0 \end{array} \\ \text{Alice} & \begin{array}{c} p=1 \\ p=0 \end{array} & \begin{bmatrix} (\alpha', \alpha') & (\beta', \gamma') \\ (\gamma', \beta') & (\alpha', \alpha') \end{bmatrix} \end{array} \tag{19}$$

where

$$\alpha' = \frac{5\alpha + 5\beta + 6\gamma}{16} \qquad \beta' = \frac{5\alpha + \beta + 10\gamma}{16} \qquad \gamma' = \frac{\alpha + 5\beta + 10\gamma}{16} \tag{20}$$

with $\alpha' > \beta' > \gamma'$. On the other hand, the quantized version of Marinatto and Weber [5] can be represented by the following payoff matrix:

$$\begin{array}{ccc} & & \text{Bob} \\ & & \begin{array}{cc} q=1 & q=0 \end{array} \\ \text{Alice} & \begin{array}{c} p=1 \\ p=0 \end{array} & \begin{bmatrix} \left(\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right) & (\gamma, \gamma) \\ (\gamma, \gamma) & \left(\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right) \end{bmatrix}. \end{array} \tag{21}$$

In comparison with the classical version payoff matrix, i.e. equation (1), both Marinatto and Weber's payoff matrix (21) and our payoff matrix (19) show a clear advantage over the classical version as the payoffs for the two players are the same for the two pure NE in the quantum versions of the game. Hence there is no incentive for the two players to prefer one NE over the other. However, as pointed out by Benjamin [6], in Marinatto's quantum version, in the absence of any communication the two players could inadvertently end up with mismatched strategies, i.e., (1, 0) or (0, 1) which corresponds to a minimum possible payoff, i.e., γ, for both the players. It is important to note that in our version of the quantum battle of sexes the payoffs corresponding to the worst-case situation are different for the two players. This particular feature leads to a unique solution for the game by providing a straightforward reason for rational players to go for one of the NE, i.e. (1, 1) for the particular choice of parameters given by equation (14).

It can be seen from the payoff matrix (19) that the payoff for the two players is maximum for the two NE, (0, 0) and (1, 1), but for Alice the rational choice is $p^* = 1$ since her payoff is maximum, i.e. α', when Bob decides to play $q^* = 1$ and equals β' if Bob decides to play $q^* = 0$, which is higher than the worst possible payoff, i.e. γ'. In a similar manner for Bob the rational choice is $q^* = 1$ since his payoff is maximum, i.e. α', when Alice also plays $p' = 1$ and equals β' when Alice plays $p' = 0$ which is better than the worst possible. Thus for the initial quantum state with parameters given by equation (14), NE (1, 1) is clearly a preferred strategy for both the players giving a unique solution to the game. Similarly an initial quantum state, for example, with parameters $|a|^2 = |d|^2 = |c|^2 = \frac{5}{16}$, $|b|^2 = \frac{1}{16}$ can be found for which (0, 0) is a preferred strategy pair for both the players giving a unique solution for the game.

3. Summary

We analysed the game of quantum battle of sexes using the approach developed by Marinatto and Weber [5]. Instead of restricting to the maximally entangled initial quantum state we considered a general initial quantum state. Exploiting the additional parameters in the initial state we presented a condition for which unique solution of the game can be obtained. In particular, we addressed the issues pointed out by Benjamin [6] in Marinatto and Weber's [5] quantum version of the battle of sexes game. In our approach the difference in the payoffs for the two players corresponding to the so-called worst-case situation leads to a unique solution of the game. Our results reduce to that of Marinatto and Weber under appropriate conditions.

Appendix

A.1. Derivation of initial density matrix

The initial quantum state, i.e. equation (2), can be used to obtain the required initial density matrix

$$\rho_{\mathrm{in}} = (a|OO\rangle + b|OT\rangle + c|TO\rangle + d|TT\rangle)(a^*\langle OO| + b^*\langle TO| + c^*\langle OT| + d^*\langle TT|) \tag{A.1}$$

where $*$ stands for complex conjugate. The density matrix can also be written in the following form:

$$\begin{aligned}\rho_{\mathrm{in}} = {} & |a|^2|OO\rangle\langle OO| + ab^*|OO\rangle\langle TO| + ac^*|OO\rangle\langle OT| + ad^*|OO\rangle\langle TT| \\ & + ba^*|OT\rangle\langle OO| + |b|^2|OT\rangle\langle TO| + bc^*|OT\rangle\langle OT| + bd^*|OT\rangle\langle TT| \\ & + ca^*|TO\rangle\langle OO| + cb^*|TO\rangle\langle TO| + |c|^2|TO\rangle\langle OT| + cd^*|TO\rangle\langle TT| \\ & + da^*|TT\rangle\langle OO| + db^*|TT\rangle\langle TO| + dc^*|TT\rangle\langle OT| + |d|^2|TT\rangle\langle TT|.\end{aligned} \tag{A.2}$$

A.2. Calculation for the payoffs

From equation (4) the final density operator is

$$\begin{aligned}\rho_f = {} & pqI_A \otimes I_B\rho_{\mathrm{in}}I_A^\dagger \otimes I_B^\dagger + p(1-q)I_A \otimes C_B\rho_{in}I_A^\dagger \otimes C_B^\dagger + q(1-p)C_A \otimes I_B\rho_{\mathrm{in}}C_A^\dagger \otimes I_B^\dagger \\ & + (1-p)(1-q)C_A \otimes C_B\rho_{\mathrm{in}}C_A^\dagger \otimes C_B^\dagger.\end{aligned} \tag{A.3}$$

With the help of equation (3) ρ_f becomes

$$\begin{bmatrix} pq\zeta + |d|^2 & pq\epsilon + dc^* & pq\omega + db^* & pq\xi + da^* \\ +p(|b|^2 - |d|^2) & +p(ba^* - dc^*) & +p(ba^* - db^*) & +p(cb^* - da^*) \\ +q(|c|^2 - |d|^2) & +q(cd^* - dc^*) & +q(ca^* - db^*) & +q(cb^* - da^*) \\ -pq\epsilon + cd^* & -pq\zeta + |c|^2 & -pq\xi + cb^* & -pq\omega + ca^* \\ +p(ab^* - cd^*) & +p(|a|^2 - |c|^2) & +p(ad^* - cb^*) & +p(ac^* - ca^*) \\ +q(dc^* - cd^*) & +q(|d|^2 - |c|^2) & +q(da^* - cb^*) & +q(db^* - ca^*) \\ -pq\omega + bd^* & -pq\xi + bc^* & -pq\zeta + |b|^2 & -pq\epsilon + ba^* \\ +p(db^* - bd^*) & +p(da^* - bc^*) & +p(|d|^2 - |b|^2) & +p(dc^* - ba^*) \\ +q\,(ac^* - bd^*) & +q(ad^* - bc^*) & +q(|a|^2 - |b|^2) & +q(ab^* - ba^*) \\ pq\xi + ad^* & pq\omega + ac^* & pq\epsilon + ab^* & pq\zeta + |a|^2 \\ +p(cb^* - ad^*) & +p(ca^* - ac^*) & +p(cd^* - ab^*) & +p(|c|^2 - |a|^2) \\ +q(bc^* - ad^*) & +q(bd^* - ac^*) & +q(ba^* - ab^*) & +q(|b|^2 - |a|^2) \end{bmatrix}$$

where we defined

$$\epsilon = ab^* - ba^* - cd^* + dc^* \qquad \zeta = |a|^2 - |b|^2 - |c|^2 + |d|^2$$
$$\omega = ac^* - bd^* - ca^* + db^* \qquad \xi = ad^* - bc^* - cb^* + da^*$$

and the payoff functions (the mean values of these operators, i.e. $\$_A(p,q) = \mathrm{Tr}[P_A\rho_f]$ and $\$_B(p,q) = \mathrm{Tr}[P_B\rho_f]$) using equaitons (A.3), (5), (6) become

$$\begin{aligned}\$_A(p,q) = {} & p[q(\alpha+\beta-2\gamma)(|a|^2-|b|^2-|c|^2+|d|^2)+(\gamma-\beta)|a|^2+(\alpha-\gamma)|b|^2\\ & +(\beta-\gamma)|c|^2+(\gamma-\alpha)|d|^2]+q[(\gamma-\beta)|a|^2+(\beta-\gamma)|b|^2+(\alpha-\gamma)|c|^2\\ & +(\gamma-\alpha)|d|^2]+\alpha|d|^2+\gamma|c|^2+\gamma|b|^2+\beta|a|^2\end{aligned} \tag{A.4}$$

$$\begin{aligned}\$_B(p,q) = {} & q[p(\alpha+\beta-2\gamma)(|a|^2-|b|^2-|c|^2+|d|^2)+(\gamma-\alpha)|a|^2+(\alpha-\gamma)|b|^2\\ & +(\beta-\gamma)|c|^2+(\gamma-\beta)|d|^2]+p[(\gamma-\alpha)|a|^2+(\beta-\gamma)|b|^2+(\alpha-\gamma)|c|^2\\ & +(\gamma-\beta)|d|^2]+\beta|d|^2+\gamma|c|^2+\gamma|b|^2+\alpha|a|^2.\end{aligned} \tag{A.5}$$

The above equations lead to equations (7), (8) in terms of quantities defined by equation (9).

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